PP36712 and PP36713. Proposed by Pirkuliyev Rovsen.

If $x \in \left(0, \frac{\pi}{2}\right)$, then prove:

a) $\cos x\left(2^{\frac{1}{\sin x}} - 1\right) > 2^{\cot x} - 1;$

b) $\sin x\left(2^{\frac{1}{\cos x}} - 1\right) > 2^{\tan x} - 1.$

Solution by Arkady Alt, San Jose, California, USA.

Solution 1.

First we will prove that for any $a, b \in (0,1)$ holds inequality

(**1**) $a(2^{1/b} - 1) > 2^{a/b} - 1.$

Let $t := 2^{1/b}$. Then $t > 1$ and inequality (**1**) becomes $a(t-1) > t^a - 1.$

By Mean Value Theorem there is $c \in (1, \infty)$ such that $\frac{t^a - 1}{t - 1} = ac^{a-1}.$

Since $a - 1 < 0$ and $c > 1$ then $ac^{a-1} < a$ and, therefore, $\frac{t^a - 1}{t - 1} < a \Leftrightarrow$
$a(t-1) > t^a - 1.$

By replacing (a, b) in inequality (**1**) with $(\cos x, \sin x)$ and $(\sin x, \cos x)$ we obtain, respectively, inequalities in **a**) and **b**).

Solution 2.(Without Mean Value Therem).

By replacing in Bernoulli 2 Inequality $(1+x)^p \leq 1 + px,\ x > -1, p \in (0,1]$ (x,p) with $(t-1, a)$ we obtain $t^a = (1 + (t-1))^a < 1 + a(t-1) \Leftrightarrow$
$t^a - 1 < a(t-1)$ (because equality occurs iff $t = 1$ or $a = 1$ but in our case $t - 1 > 0$ and $0 < a < 1$),

Also, by Weighted AM-GM Inequality

$1 + a(t-1) = 1 - a + at > 1^{1-a} \cdot t^a = t^a \Leftrightarrow a(t-1) > t^a - 1$

(not $\geq$ by the same reason as above).